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LETTER TO THE EDITOR

On the Rogers–Szegő polynomials

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Abstract. An orthogonality relation on the full real line for the Rogers–Szegő polynomials is discussed. It is argued that Fourier transformation with the standard exponential kernel $\exp(ixy)$ relates the Rogers–Szegő and Stieltjes–Wigert functions.

Here we study some properties of the Rogers–Szegő polynomials and, in particular, their connection with the Stieltjes–Wigert polynomials. The knowledge of these properties may be useful in constructing concrete realizations for the q-oscillator wavefunctions, expressed in terms of orthogonal q-polynomials and their weight functions [1–7].

As is known [8, 9], the Rogers–Szegő polynomials are defined as

$$H_n(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \qquad 0 < q < 1 \tag{1}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

and $(a,q)_n = \prod_{j=0}^{n-1}(1-aq^j)$. They satisfy the orthogonality relation on the unit circle [8, 10]

$$\frac{1}{2\pi i} \oint_{|w|=1} H_m\left(-\frac{w^*}{\sqrt{q}};q\right) H_n\left(-\frac{w}{\sqrt{q}};q\right) \vartheta_3\left(\frac{\log w}{2i} q^{1/2}\right) \frac{dw}{w} = \frac{(q;q)_m}{q^m}\delta_{mn} \tag{2}$$

where $\vartheta_3(z,q)$ is the theta-function, i.e.,

$$\vartheta_3(z,q) \equiv \vartheta_3(z|\tau) = \sum_{j=-\infty}^{\infty} q^{j^2} e^{2ijz} \tag{3}$$

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and $q = \exp(\pi i\tau)$ (see, for example, [11]). The Rogers–Szegő polynomials can be expressed through the q-Hermite polynomials $H_n(x|q)$ [12] as

$$H_n(-e^{2ix}; q) = i^{-n} e^{inx} H_n(\sin x|q). \tag{4}$$

The Rogers–Szegő polynomials satisfy the orthogonality relation on the full real line

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_m(-q^{-1/2}e^{-2i\kappa x}; q) H_n(-q^{-1/2}e^{2i\kappa x}; q) e^{-x^2}\, dx = \frac{(q;q)_m}{q^m}\delta_{mn} \tag{5}$$

where $q = \exp(-2\kappa^2)$. To prove (5), make the substitution $w = e^{i\alpha}$ $(0 \leqslant \alpha < 2\pi)$ and use the modular transformation [11]

$$\vartheta_3(z|\tau) = \frac{e^{-iz^2/\pi\tau}}{\sqrt{-i\tau}} \vartheta_3(z\tau^{-1}| - \tau^{-1}) \tag{6}$$

to rewrite the left-hand side of (2) as

$$\frac{1}{2\sqrt{\pi}\kappa} \int_0^{2\pi} d\alpha\, H_m(-q^{-1/2}e^{-i\alpha}; q) H_n(-q^{-1/2}e^{i\alpha}; q) \vartheta_3\left(\frac{\pi i\alpha}{2\kappa^2}, e^{-\pi^2/\kappa^2}\right) e^{-\alpha^2/4\kappa^2}. \tag{7}$$

Using expansion (3) and taking into account the uniform convergence of this series in any bounded domain of values of z [11], substitute (3) into (7) and integrate it termwise. This gives

$$\frac{1}{2\sqrt{\pi}\kappa} \sum_{j=-\infty}^{\infty} e^{-\pi^2 j^2/\kappa^2} \int_0^{2\pi} d\alpha\, H_m(-q^{-1/2}e^{-i\alpha}; q) H_n(-q^{-1/2}e^{i\alpha}; q) \exp\left(-\frac{\alpha^2}{4\kappa^2} - \frac{\pi\alpha j}{\kappa^2}\right). \tag{8}$$

The change of variable $2\kappa x_j = \alpha + 2\pi j$, $(\pi/\kappa)j = x_j^{\min} \leqslant x_j \leqslant x_j^{\max} = (\pi/\kappa)(j+1)$ allows one to sum (8) with respect to j and leads to the left-hand side of (5), if one takes into account that $x_{j-1}^{\max} = x_j^{\min}$.

The more general case of (5), i.e., orthogonality of the Askey–Wilson polynomials [13] with respect to a Ramanujan-type measure is discussed in [14].

From (5), it follows that in analogy with the classical Hermite functions (or the linear harmonic-oscillator wavefunction in quantum mechanics) $H_n(x)\exp(-x^2/2)$ [15], the orthonormalized system

$$\psi_n^{\mathrm{RS}}(x; q) = \frac{q^{n/2}}{\pi^{1/4}(q;q)_n^{1/2}} H_n(-q^{-1/2}e^{2i\kappa x}; q) \exp\left(\frac{-x^2}{2}\right) \tag{9}$$

can be called the Rogers–Szegő functions. We remark that to prove the completeness of (9) in L^2, the bilinear kernel

$$K_z(x, y; q) = \sum_{n=0}^{\infty} z^n \psi_n^{\mathrm{RS}}(x; q) \psi_n^{\mathrm{RS}}(y; q)^* \qquad |z| < 1 \tag{10}$$

where $*$ denotes the complex conjugate, needs to be introduced (for details see [15, 16]). Due to the orthonormality of (9), this kernel has the important reproducing properties

$$\int_{-\infty}^{\infty} K_z(x, y; q)\psi_n^{\mathrm{RS}}(y; q)\,\mathrm{d}y = z^n \psi_n^{\mathrm{RS}}(x; q) \tag{11}$$

$$\int_{-\infty}^{\infty} K_z(x, t; q)K_{z'}(t, y; q)\,\mathrm{d}t = K_{zz'}(x, y; q). \tag{12}$$

The explicit form of (10) is found by taking into account relation (4) and using the bilinear generating function (or the Poisson kernel) for the q-Hermite polynomials $H_n(x|q)$ [12], i.e.,

$$K_z(x, y; q) = \frac{1}{\sqrt{\pi}} \frac{(qz^2 \mathrm{e}^{2\mathrm{i}\kappa(x-y)}; q)_\infty \exp(-x^2 + y^2/2)}{(qz, z\mathrm{e}^{2\mathrm{i}\kappa(x-y)}, -q^{1/2}z\mathrm{e}^{2\mathrm{i}\kappa x}, -q^{1/2}z\mathrm{e}^{-2\mathrm{i}\kappa y}; q)_\infty} \tag{13}$$

where $(z_1, \ldots, z_k; q)_\infty = \prod_{j=1}^{k}(z_j; q)_\infty$. For a more detailed discussion of the completeness property in the case similar to (9) of an orthogonal system of functions, we refer the reader to [17].

To evaluate a Fourier transform of the Rogers–Szegő functions, we remember that the Stieltjes–Wigert polynomials $s_n(x)$ are defined as (see, for example, [18])

$$s_n(x) = (-1)^n q^{2n+1/4}(q; q)_n^{-1/2} \tilde{s}_n(-q^{1/2}x; q) \qquad 0 < q < 1 \tag{14}$$

where, in analogy with (1), the notation

$$\tilde{s}_n(x; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} x^k \tag{15}$$

is used. These polynomials satisfy the orthogonality relation

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \tilde{s}_m(-q^{-1/2}\mathrm{e}^{-2\kappa x}; q)\tilde{s}_n(-q^{-1/2}\mathrm{e}^{-2\kappa x}; q)\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{(q; q)_m}{q^m}\delta_{mn} \tag{16}$$

and can be expressed through the continuous q^{-1}-Hermite polynomials $h_n(x|q) = \mathrm{i}^{-n} H_n(\mathrm{i}x|q^{-1})$ [19], i.e.,

$$\tilde{s}_n(-q^{-n}\mathrm{e}^{-2x}; q) = \mathrm{e}^{-nx} h_n(\sinh x|q). \tag{17}$$

Besides, as follows from (1) and (15),

$$\tilde{s}_n(x; q^{-1}) = H_n(xq^{-n}; q).$$

To prove that for arbitrary complex a, the Rogers–Szegő $H_n(a\mathrm{e}^{2\mathrm{i}\kappa x}; q)\exp(-x^2/2)$ and the Stieltjes-Wigert $\tilde{s}_n(a\mathrm{e}^{-2\kappa y}; q)\exp(-y^2/2)$ functions are related to each other by the Fourier transformation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(a\mathrm{e}^{2\mathrm{i}\kappa x}; q)\mathrm{e}^{\mathrm{i}xy - x^2/2}\,\mathrm{d}x = \tilde{s}_n(a\mathrm{e}^{-2\kappa y}; q)\mathrm{e}^{-y^2/2} \tag{18}$$

substitute the explicit representation (1) into the left-hand side of (18) and integrate over x with the aid of the formula

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy-x^2/2}\,dx = e^{-y^2/2}. \tag{19}$$

As follows from (18) and its inverse transformation, the relation between the Rogers–Szegő and the Stieltjes–Wigert polynomials can be also written in terms of the difference-differentation formulae ($\partial_x = d/dx$)

$$\begin{aligned} H_n(ae^{-2i\kappa x}; q) &= e^{x^2/2}\tilde{s}_n(ae^{2i\kappa\partial_x}; q)e^{-x^2/2} \\ \tilde{s}_n(ae^{-2\kappa x}; q) &= e^{x^2/2}H_n(ae^{2\kappa\partial_x}; q)e^{-x^2/2} \end{aligned} \tag{20}$$

which are q-analogues of the relation [20]

$$H_n(x) = i^n e^{x^2/2}H_n(i\partial_x)e^{-x^2/2} \tag{21}$$

for the classical Hermite polynomials $H_n(x)$. To verify (20), one needs just the evident relations

$$e^{2n\kappa\partial_x}e^{ixy} = e^{2in\kappa y}e^{ixy} \qquad n = 0, 1, 2, \ldots.$$

Observe also the following limit formulae (cf [21]):

$$\lim_{q\to 1^-}\left(\frac{q}{1-q}\right)^{n/2} H_n(-q^{-1/2}e^{2i\kappa x}; q) = (i\sqrt{2})^{-n}H_n(x) \tag{22}$$

$$\lim_{q\to 1^-}\left(\frac{q}{1-q}\right)^{n/2} \tilde{s}_n(-q^{-1/2}e^{-2\kappa x}; q) = 2^{-n/2}H_n(x) \tag{23}$$

where $q = \exp(-2\kappa^2)$. To prove (22), use the binomial theorem for the quantities $u = e^{i\kappa\partial_x}$ and $v = -q^{-1/2}e^{2i\kappa x}$, satisfying the commutation relation of the quantum plane $uv = qvu$, i.e. (see, for example, [18]),

$$(u+v)^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q v^k u^{n-k}$$

to define the q-raising operator $b^+(x; q)$ as (cf [1, 5])

$$\begin{aligned} H_n(-q^{-1/2}e^{2i\kappa x}; q) &= (-q^{-1/2}e^{2i\kappa x} + e^{i\kappa\partial_x})^n * 1 \\ &= \left(-i\sqrt{\frac{1-q}{q}}\right)^n e^{x^2/2}[b^+(x; q)]^n e^{-x^2/2}. \end{aligned} \tag{24}$$

Since, in the limit when $q \to 1^-$ (or $\kappa \to 0$), the operator

$$b^+(x; q) = \frac{e^{i\kappa x}}{i\sqrt{1-q}}(e^{i\kappa x} - q^{3/4}e^{i\kappa\partial_x})$$

coincides with the harmonic-oscillator creation operator $a^+(x) = \frac{1}{\sqrt{2}}(x - \partial_x)$, it follows from (24) that the left-hand side of (22) in this limit tends to

$$e^{x^2/2}\left[\sqrt{2}a^+(x)\right]^n e^{-x^2/2} = H_n(x).$$

Now, it is easily verified that (18) and (22) yield the limit formula (23).

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